Game THEORY

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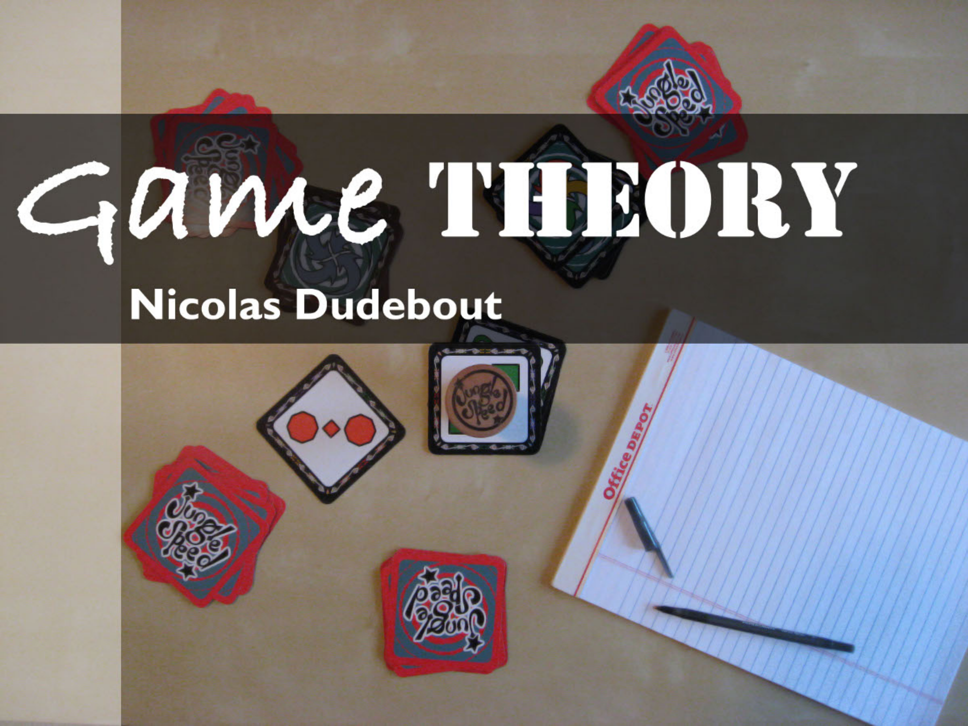

Previously in *Game Theory*

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- decision makers:
  - choices
  - preferences

Previously in *Game Theory*

- decision makers:
  - choices
  - preferences
- solution concepts:
  - best response
  - Nash equilibrium

Rock, paper, scissors

Rock, paper, scissors

	R	P	S
R	$0, 0$	$-1, 1$	$1, -1$
P	$1, -1$	$0, 0$	$-1, 1$
S	$-1, 1$	$1, -1$	$0, 0$

Learning in games

Learning in games
Repeated games

Learning in games

Best Response learning

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1. Guess what the opponent(s) will play

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2. Play a Best Response to that guess

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3. Observe the play

Best Response learning

1. Guess what the opponent(s) will play
2. Play a Best Response to that guess
3. Observe the play
4. Update the guess

BR learning: Cournot dynamics

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Guess = last action played

BR learning: Cournot dynamics

Guess = last action played

	C	D
C	$2, 2$	$-1, 3$
D	$3, -1$	$0, 0$

BR learning: Cournot dynamics

Guess = last action played

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Guess $=$ empirical distribution of play

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BR learning: Fictitious play

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	L	C	R
U	$0, 0$	$0, 1$	$1, 0$
M	$1, 0$	$0, 0$	$0, 1$
D	$0, 1$	$1, 0$	$0, 0$

Evolutionary learning

Evolutionary learning

Action set: A
Utility function: u

Evolutionary learning

Action set: A
Utility function: u

$p \in \Delta(A), k \in A$
$\dot{p}_k = p_k \left(u(k,p) - u(p,p)\right)$

Battle of the Sexes

Battle of the Sexes

	O	F
O	$3, 2$	$0, 0$
F	$0, 0$	$2, 3$

Correlated equilibrium (CE)

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$a^* \in A = \prod_i A_i$ is a NE:

$$\forall i, \forall a'_i, u_i(a^*_i, a^*_{-i}) \geq u_i(a'_i, a^*_{-i})$$

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$\alpha \in \prod_i \Delta(A_i)$ is a NE: $\forall i, \forall a_i, \forall a_i',$

$$\sum_{a_{-i}} u_i(a_i, a_{-i})\alpha(a) \geq \sum_{a_{-i}} u_i(a_i', a_{-i})\alpha(a)$$

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$$\sum_{a_{-i}} u_i(a_i, a_{-i})\alpha(a) \geq \sum_{a_{-i}} u_i(a_i', a_{-i})\alpha(a)$$

$\pi \in \Delta(A)$ is a CE: $\forall i, \forall a_i, \forall a_i',$

$$\sum_{a_{-i}} u_i(a_i, a_{-i})\pi(a) \geq \sum_{a_{-i}} u_i(a_i', a_{-i})\pi(a)$$

No regret learning

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$$u_i(k, a_{-i}) - u_i(j, a_{-i})$$

No regret learning

$$u_i(k, a_{-i}) - u_i(j, a_{-i})$$

$$R^i_{jk}(t) = \sum_{\tau=0:a_i(\tau)=j}^{t} u_i(k, a_{-i}(\tau)) - u_i(j, a_{-i}(\tau))$$

No regret learning

$$u_i(k, a_{-i}) - u_i(j, a_{-i})$$

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Regret matching converges to the correlated equilibria set.

Learning in games

Learning in games

- Best response

Learning in games

- Best response
- Replicator dynamics

Learning in games

- Best response
- Replicator dynamics
- No regret

Repeated games

Markov Decision Process (MDP)

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state space X

action space U

transition $P : X \times U \to \Delta(X)$

reward $r : X \times U \to \mathbb{R}$

discount factor $\delta \in [0, 1]$

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transition $P : X \times U \to \Delta(X)$

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discount factor $\delta \in [0, 1]$

$$U(x(\cdot), u(\cdot)) = \sum_{t=0}^{+\infty} \delta^t r(x(t), u(t))$$

MDP (continued)

history $\mathcal{H} \in \prod(X, U)$
policy $\pi : \mathcal{H} \to \Delta(U)$

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history $\mathcal{H} \in \prod(X, U)$
policy $\pi : \mathcal{H} \to \Delta(U)$
$V^\pi(x_0) = \mathbb{E}_\pi \left[U(x(\cdot), u(\cdot)) \right]$

$$V(x_0) = \max_\pi V^\pi(x_0)$$

Principle of Optimality

Bellman's equation:

$$V(x_0) = \max_{u_0} \left[r(x_0, u_0) + \delta V(P(x_0, u_0)) \right]$$

Dynamic Programming

Solving the MDP:

Dynamic Programming

Solving the MDP:

- knowing P: value iteration

Dynamic Programming

Solving the MDP:

- knowing P: value iteration
- not knowing P: online learning

Repeated game

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Game $(\mathcal{I}, \prod_i A_i, \prod_i u_i)$

Repeated game

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Discount factor δ

$$U_i(a(\cdot)) = \sum_{t=0}^{+\infty} \delta^t u_i(a(t))$$

Repeated game

Game $(\mathcal{I}, \prod_i A_i, \prod_i u_i)$
Discount factor δ

$$U_i(a(\cdot)) = \sum_{t=0}^{+\infty} \delta^t u_i(a(t))$$

Strategy $\sigma : \mathcal{H} \to \prod_i \Delta(A_i x)$

$$V_i(\sigma) = \mathbb{E}_\sigma \left[U_i(a(\cdot))\right]$$

Nash equilibrium

Player i:

- choices σ_i
- utility V_i

Nash equilibrium

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Nash equilibrium is not strong enough!
(Explanation on the whiteboard $\Rightarrow$)

Information structure

Information structure

- perfect
- imperfect

Information structure

- perfect
- imperfect

- public
- private (beliefs)

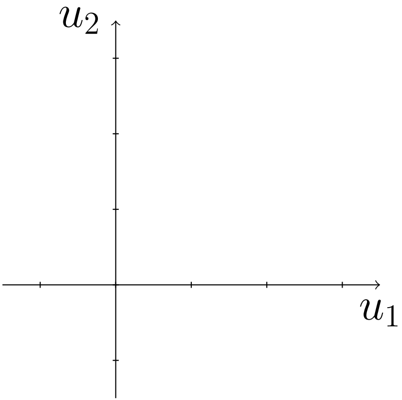
Folk theorem

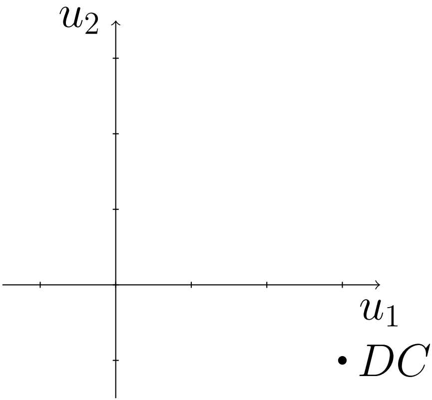
Any feasible, strictly individually rational payoff can be sustained by a sequentially rational equilibrium.

Folk theorem

Any feasible, strictly individually rational payoff can be sustained by a sequentially rational equilibrium.

Holy grail for repeated games.


u_2
u_1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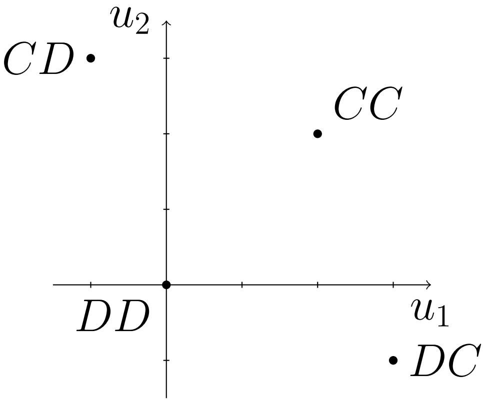
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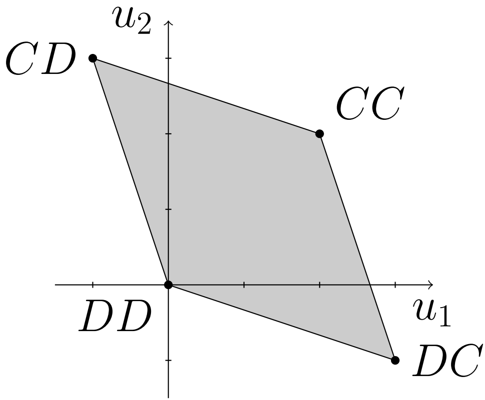

u_2
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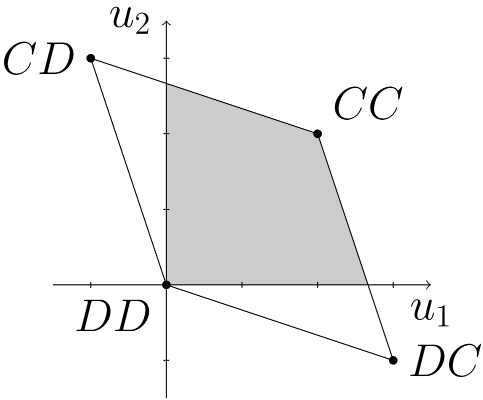
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DD
u_1
DC

Research

Weakly belief-free equilibria

Characterization of repeated games with correlated equilibria.

Repeated games

Repeated games

- Dynamic programming

Repeated games

- Dynamic programming
- Repeated games

Repeated games

- Dynamic programming
- Repeated games
- Folk theorem

Learning in games

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Questions, Comments