Empirical-evidence Equilibria in Stochastic Games

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Asset management on the stock market

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Single-agent Setup

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$$\max_{\sigma} \mathbb{E}_{\sigma}\left[\sum_{t=0}^{\infty} \delta^t u\left(x^t, a^t, s^t\right)\right]$$

Single-agent Setup

$$\begin{tikzcd} \boxed{\begin{array}{c} x^+ \sim f(x, a, s) \\ a \sim \sigma(h) \end{array}} & \boxed{\begin{array}{c} w^+ \sim n(w, x, a) \\ s \sim \nu(w) \end{array}} \arrow[l, "s"'] \end{tikzcd}$$

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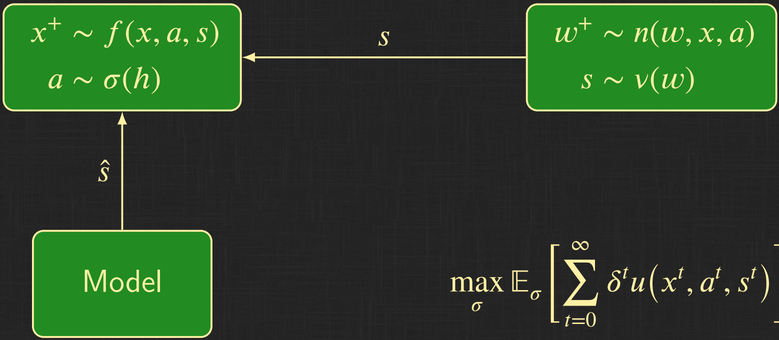
Single-agent Setup



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Single-agent Setup

$$\begin{tikzcd}
\boxed{\begin{array}{c} x^+ \sim f(x, a, s) \\ a \sim \sigma(x, z) \end{array}} & \boxed{\begin{array}{c} w^+ \sim n(w, x, a) \\ s \sim \nu(w) \end{array}} \arrow[l, "s"'] \\
\boxed{\begin{array}{c} z^+ \sim m(z) \\ \hat{s}^+ \sim \mu(z) \end{array}} \arrow[u, "\hat{s}"] &
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$$\max_{\sigma} \mathbb{E}_{\sigma}\left[\sum_{t=0}^{\infty} \delta^t u\left(x^t, a^t, s^t\right)\right]$$

- μ consistent with σ
- σ optimal w.r.t. μ

Depth-k Consistency

Binary stochastic process s
0100010001001010010110111010000111010101...

- 0 characteristic: $\mathbb{P}[s=0], \mathbb{P}[s=1]$
- 1 characteristic: $\mathbb{P}\left[ss^+=00\right], \mathbb{P}\left[ss^+=10\right],$ $\mathbb{P}\left[ss^+=01\right], \mathbb{P}\left[ss^+=11\right]$
- ...
- k characteristic: probability of strings of length $k+1$

Definition

Two processes s and $\hat{s}$ are depth-k consistent if they have the same k characteristic

Complete Picture

$$
\begin{tikzcd}
\boxed{\begin{array}{c} x^+ \sim f(x, a, s) \\ a \sim \sigma(x, z) \end{array}} & \boxed{\begin{array}{c} w^+ \sim n(w, x, a) \\ s \sim \nu(w) \end{array}} \arrow[l, "s"'] \\
\boxed{\begin{array}{c} z^+ \sim m_k(z) \\ \hat{s} \sim \mu(z) \end{array}} \arrow[u, "\hat{s}"] &
\end{tikzcd}
$$

Fix a depth $k \in \mathbb{N}$

z contains the last k observed signals

$$
\mu\big(z = (s_1, s_2, \dots, s_k)\big)[s_{k+1}] = \mathbb{P}_\sigma\left[s^{t+1} = s_{k+1} \,\middle|\, s^t = s_k, \dots, s^{t-k+1} = s_1\right]
$$

Empirical-evidence Optimality

Definition

(σ, μ) is an empirical-evidence optimum (EEO) for k iff

- σ is optimal w.r.t. μ
- μ is depth-k consistent with σ

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- μ is depth-k consistent with σ

Existence Result

Theorem

For all k and ϵ, there exists an ϵ EEO for k

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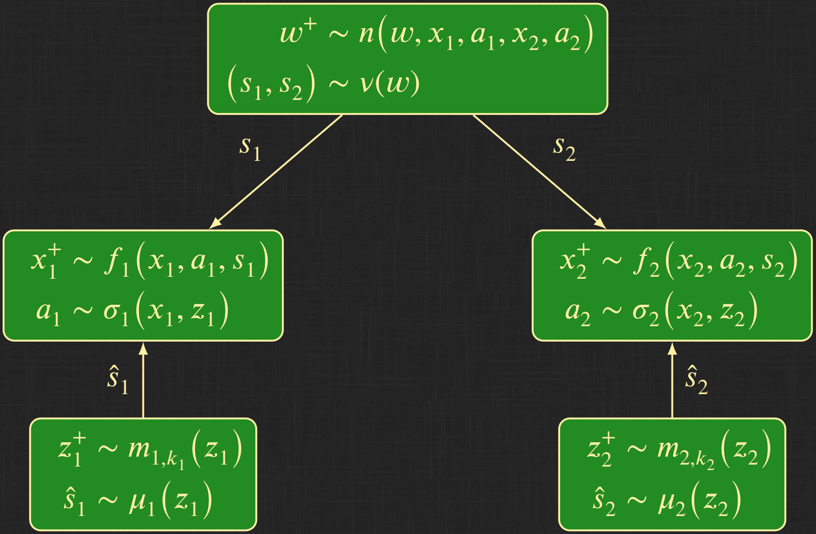
Proof sketch

- Technical assumption insures ergodicity of s
- $T : \sigma \xmapsto{consistency} \mu \xmapsto{\epsilon\ optimality} \sigma$ is continuous
- $\sigma : \mathscr{X} \times \mathscr{Z} \to \Delta(\mathscr{A})$ is parametrized over a simplex
- Apply Brouwer's fixed point theorem to T

Multiagent Setup



Empirical-evidence Equilibrium

$$\text{Strategies}\ \ \sigma = \left(\sigma_1, \sigma_2, \ldots, \sigma_N\right)$$
$$\text{Models}\ \ \mu = \left(\mu_1, \mu_2, \ldots, \mu_N\right)$$
$$\text{Depths}\ \ k = \left(k_1, k_2, \ldots, k_N\right)$$

Definition

(σ, μ) is an empirical-evidence equilibrium (EEE) for k iff

- for all i, σ_i is optimal w.r.t. μ_i
- for all i, μ_i is depth-k_i consistent with σ

Theorem

For all k and ϵ, there exists an ϵ EEE for k

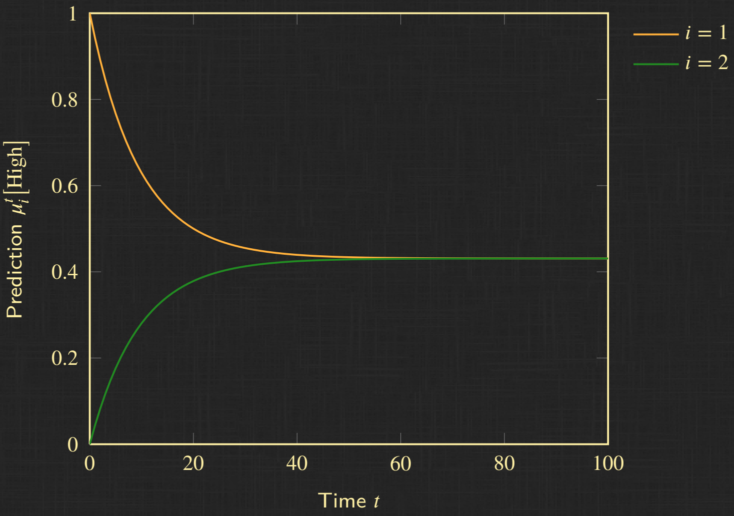
Learning Setup

State	holdings $x_i \in \{0 .. M\}$
Action	sell one, hold, or buy one $a_i \in \{-1, 0, 1\}$
Signal	price $p \in \{\text{Low}, \text{High}\}$
Dynamic	$x_i^+ = x_i + a_i$
Stage cost	$p \cdot a_i$
Nature	market trend $b \in \{\text{Bull}, \text{Bear}\}$
	$w = (b, p)$

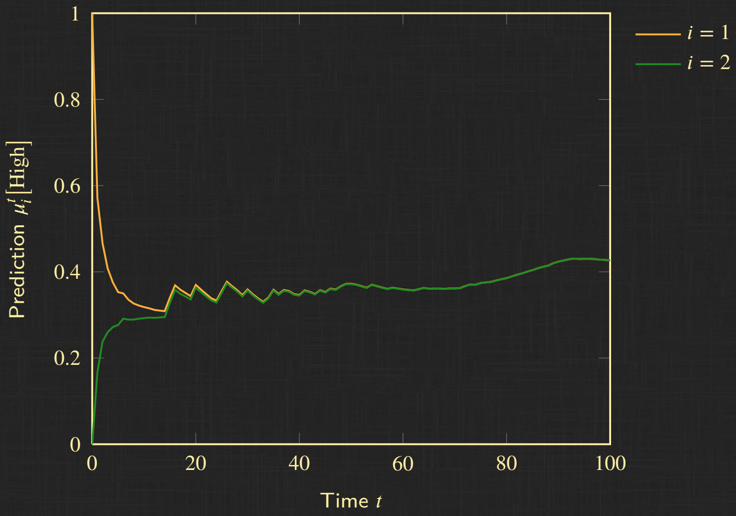
0. Pick arbitrary depth-0 models μ
1. Design strategies σ optimal w.r.t. models μ
2. Formulate consistent models μ_{upd}, then, back to 1.

$$\mu_i^{t+1} = (1-\alpha)\mu_i^t + \alpha\left(\mu_{i,\text{upd}}^t - \mu_i^t\right)$$

Learning Results: Offline



Learning Results: Online



Concluding Remarks

Comparison with mean-field equilibria

- Identical agents with a specific signal
- Depth-0 model
- Large number of agents to recover Nash equilibrium

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Future directions

- Endogenous model $\left(z^+ \sim m(z, x, a)\right)$
- Quality of EEEs
- Learning EEEs